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THE CROSS-SECTION PAPER AS A MATHEMATICAL INSTRUMENT

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The following note[1] is addressed to teachers and prospective teachers of mathematics in elementary and secondary schools and colleges, who have come to recognize as fundamental the problem of closer *correlation of arithmetic, algebra, and geometry with one another and with the various domains of application*, or the problem of *unification of elementary pure and applied mathematics.* Pure mathematics is, as it were, a language for the convenient expression and investigation of relations the most diverse in ordinary life and in nature. The principles of the language are not arbitrary, but are imposed by the phenomena demanding convenient expression. The problem is solved when, and only when, we put our pupils into such a physical and intellectual environment that they learn to see and to think the mathematics for and of themselves.

In this note I wish to suggest possibilities, which may have escaped attention, in the systematic use of cross-section paper[2] as a unifying

[1]Read in part March 5, 1906, before a joint meeting of the Junior Mathematical Club and the Mathematical Club of the University High School of the University of Chicago.

[2]In the few diagrams to be given here only the square-ruled paper appears. I advise the use of the various styles of ruling—into squares, into rectangles, into parallelograms, into triangles, and with concentric circles and diverging radii—obtainable, for instance, from the Atlas School Supply Co., Chicago. The interaction of the various papers is especially important.

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element in mathematics. I know of no medium serving to bring together so closely and so easily the three phases or dialects of pure mathematics—*number*, *form*, *formula*—and to lead so directly to the concept of *functionality*—a concept which since the seventeenth century has dominated advanced mathematics and the sciences; a concept which in the twentieth century, according to the auspices, will play a fundamental rôle in the reorganization of elementary mathematical education. *Functionality is the relation or (mathematical) law of connection between two or more quantities or numbers subject to simultaneous and interdependent continuous variation;* like, for instance, the relation between the simultaneous ages of a father and his son; or that between the length of the perimeter and the area of the surface of a square surrounded by a platoon of men preserving square formation and marching each directly away from a fixed flagpole; or that between the simultaneous temperature, pressure, and density of a gas under conditions varying in time or in space.

Students will gain a more easy and perfect mastery of mathematics, and their work will be full of richer direct and indirect value for them, when the primary emphasis is laid on the recognition, the depiction, and the closer study of functional relations between variable quantities. The variable quantity is the general quantity. I recognize the difficulties inherent in the notions of general and signed quantities and numbers, and in the notion of the depiction of a quantity of one kind to scale by a quantity of another kind. But in elementary teaching these difficulties should be met not primarily by the analytical processes of logic resulting in the separation of mathematics into arithmetic, algebra, geometry, only loosely related and not readily available for application. They should be met by the systematic union[1] of the three phases of mathematics in connection with problems

[1]The best modern textbooks on arithmetic show in a gratifying way that for the elementary school the problem of unification is under way to solution. The authors develop in organic relation to problems of real life, not only number work, but also to a considerable degree literal arithmetic, observational geometry, geometrical drawing, drawing to scale, plotting, and algebra.

Further, it is known that gratifying progress is making in various quarters toward the solution of the problem for the secondary school. Cf. the columns of the *School Review* and of *School Science and Mathematics*, Chicago, and J. W. A. Young's book, *The Teaching of Mathematics*, to be published soon by Longmans, Green & Co., for information as to the current movements in this country and in England, France, and Germany.

of practical and scientific nature. So approached, these fundamental conceptions are not really so difficult. Every boy loves trains, and most boys have traveled. With the cross-section paper in commonplace school use, the boy of nine or ten will readily understand and create diagrams of train motion, and will enjoy making the limited express overtake the slow freight at a certain time and place.

In this note, confining attention to the simplest algebraic and geometric aspects of functionality, I study the cross-section paper in connection with the questions of *double-entry tables* and *graphical computation*, and I hope to suggest in some measure the very central rôle which the cross-section paper as a mathematical instrument should be made to play in the unification of the elementary mathematical disciplines. By maximizing the function of the cross-section paper we secure, to speak only of pure mathematics, intense reaction between geometry and algebra. Geometry and algebra may certainly be developed independently, each with its relations to arithmetic, and no one doubts their high educational and scientific value as so developed. But this value is indeed small compared with the value to be obtained by developing them together in continuous reaction, thus releasing, as it were, abundant stores of sub-atomic energy.

The reader is requested to have at hand pencil, straight edge, and cross-section paper, and to duplicate the constructions shown in the figures, reserving for a second reading the more general parts of the text.

Double-Entry Tables—Numerical and Graphical

1. The fundamental operations of arithmetic and algebra involve two numbers, and the results of addition, etc., should be exhibited in double-entry tables, as indicated below, where for brevity the subtraction table is omitted and the given numbers X, Y are taken as positive integers at most 5. These tables, exhibiting the sum S, etc.,

$$X+Y=S\ ,\quad XY=P\ ,\quad \frac{Y}{X}=Q\ ,$$

of the two numbers X, Y are to be thought of as inscribed on square-ruled paper, each entry S or P or Q being made at the corresponding point (X, Y) of the paper. Make such tables on large sheets with entries for X and Y varying, for instance, by fifths from -10 to $+10$,

and then see clearly how $S=X+Y$, $D=Y-X$, $P=XY$, $Q=Y/X$, or any function[1] F of X and Y as a function of position of the point (X, Y) varies (in general) continuously as the point (X, Y) moves continuously.

2. Evidently the numerical double-entry tables (I, II, III) may conveniently be replaced by the graphical tables[2] (Figs. 1, 2, 3);

5	6	7	8	9	10
4	5	6	7	8	9
3	4	5	6	7	8
2	3	4	5	6	7
1 = Y	2	3	4	5	6
	$X=1$	2	3	4	5

I. Addition Table: $X+Y=S$

5	5	10	15	20	25
4	4	8	12	16	20
3	3	6	9	12	15
2	2	4	6	8	10
1 = Y	1	2	3	4	5
	$X=1$	2	3	4	5

II. Multiplication Table: $XY=P$

5	5	$2\frac{1}{2}$	$1\frac{2}{3}$	$1\frac{1}{4}$	1
4	4	2	$1\frac{1}{3}$	1	$\frac{4}{5}$
3	3	$1\frac{1}{2}$	1	$\frac{3}{4}$	$\frac{3}{5}$
2	2	1	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$
1 = Y	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
	$X=1$	2	3	4	5

III. Division Table: $\frac{Y}{X}=Q$

5	5.00	2.50	1.67	1.25	1.00
4	4.00	2.00	1.33	1.00	0.80
3	3.00	1.50	1.00	0.75	0.60
2	2.00	1.00	0.67	0.50	0.40
1 = Y	1.00	0.50	0.33	0.25	0.20
	$X=1$	2	3	4	5

III. Division Table: $\frac{Y}{X}=Q$

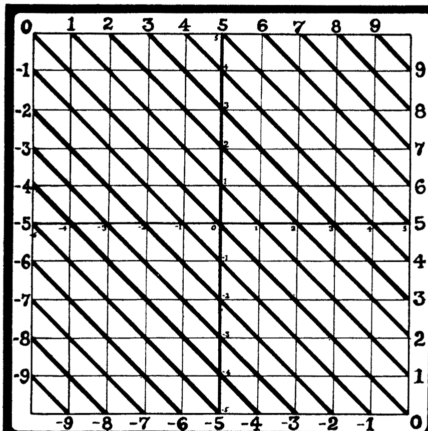
instead of making entries of the numerical values of S, etc., at individual points (X, Y) of the table, all the points (X, Y) with a certain

[1] A function F of two variable numbers X and Y is a variable number whose particular value for given values of X and Y is given or determinable in accordance with some table or formula or construction.

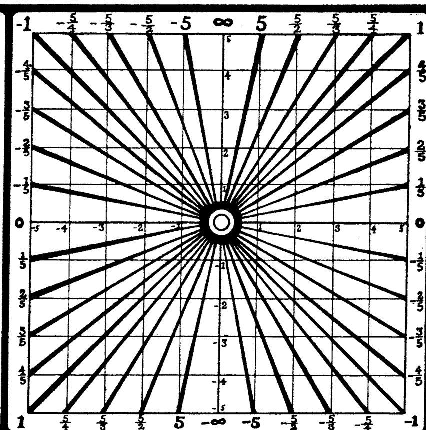
[2] The figures of this note are from drawings by Mr. J. Y. Lee, one of my pupils, to whom I express my gratitude.

entry S are joined by a curve or straight line, which bears the proper entry once for all in the margin of the table.

3. Then one comes to the notion of a function of position in the plane—the Cartesian co-ordinates X, Y offering merely one mode of

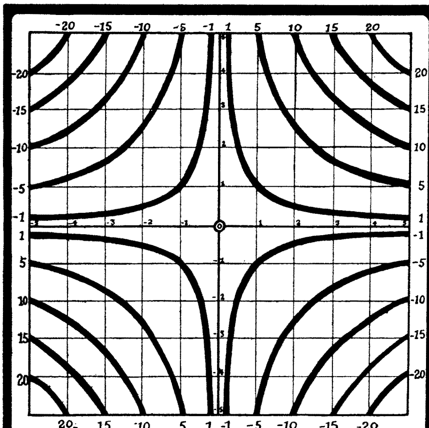


(FIG. 1) ADDITION TABLE: X+Y=S.

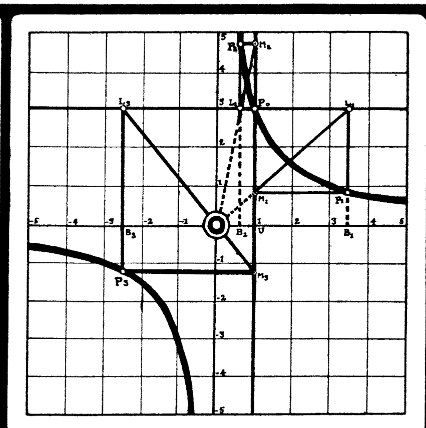


(FIG. 3.) DIVISION TABLE: $\frac{Y}{X}$=Q.

determining position, another important way being by the polar co-ordinates, the radius, and the angle of the circle-ruled paper—or even more generally of a function of position in space or on a curved



(FIG 2) MULTIPLICATION TABLE: XY=P



(FIG.4) CONSTRUCTION OF THE CURVE XY=3.

surface. These notions, of course, enter at present, and are treated thus graphically especially in work with maps, whether geographical, topographical, or meteorological. We are merely to make mathematical use of these conceptions. Thus, if the graphical addition table (Fig. 1): $X+Y=S$, is taken as the topographic map of a countryside on the horizontal XY plane, S denoting the vertical distance of the point (X, Y, S) above the XY-plane as base-plane, the countryside is a flat hillside, a plane, cutting the horizontal plane $S=0$ in the line $X+Y=0$ and passing through the point $(X, Y, S)=(1, 2, 3)$. Models in clay and other materials corresponding to these topographic maps should be constructed; in classes in physiography this is often done by cutting and pasting. The question of topographic representation of a surface with respect to various (parallel and non-parallel) base-planes should be raised, and especially in the form of the derivation of one representation from another. In this connection should be used the triangle-ruled paper, its three axes diverging at 120° from the chosen origin being taken to correspond to the three mutually rectangular axes of space.

4. Of course, a table of single entry, say for Y as a function of X, is exhibited numerically by two columns of corresponding values of X, Y, and graphically as the single curve made up of the points (X, Y) whose co-ordinates correspond.

By the systematic use of the various kinds of cross-section paper for the construction and interpretation of graphical diagrams of data of whatever origin, it is possible to make our students see and enjoy a rich and suggestive variety of useful functional relations between two or three related variables.

Graphical Computations. Linkages A, A', B

5. In geometry it is customary by elementary processes to construct to scale arithmetic expressions of the form

$$3, \ \frac{4}{5}, \ \sqrt{5}, \ 3\pm 2\sqrt{5}, \ \frac{2-3\sqrt{5}}{3-2\sqrt{5}}, \ \ldots\ ;$$

that is, of numbers arising from 1 by the sequential application to numbers already at hand of the five processes of addition, subtraction, multiplication, division and extraction of square root of a positive number—and, more generally, of such expressions arising from one

or more numbers a, b, ... supposed to be given to scale as linear segments. These constructions involve a large part of the theory of plane geometry and considerable use of the compasses. They are of limited significance, and hence relatively uninteresting.

But *introduce the element of functionality:*—for the variable number X construct and plot to scale the corresponding number Y, where Y denotes one of the various expressions in X,

$$X+3\ ,\quad 3X\ ,\quad -X+4\ ,\quad \frac{1}{X}\ ,\quad X^2\ ,\quad \sqrt{X}\ ,\quad \sqrt{1-X^2}\ ,\quad \ldots.\ ,$$

and the situation changes immediately in significance and interest.

Use, however, the *constructional methods* of the square-ruled paper, for these methods are, in fact, exceedingly simple and powerful, and of far-reaching significance in theoretical and applied mathematics. Depending on the simplest properties of similar triangles and rectangles (cf. linkages A, A', B below), properties evident after a little experience with the paper, these constructional methods should be introduced in the grades in connection with drawing and observational geometry, and held in focus throughout the high-school and college course.

Indeed, for the purposes of elementary education *our current deductive geometry is of the nature of a fetish,* to be abandoned in favor of a geometry built on a richer system of geometric axioms—a system built to recognize the cross-section paper with its wealth of intuitional relations. In this wider environment there will be abundant need and opportunity for the systematic development of logical power. The matter of the elimination of redundant axioms should be touched on only lightly in secondary-school courses.

6. The cross-section paper is at present used primarily for *plotting,* that is, for recording in graphical form tables of arithmetical data, the result of *arithmetical computations* or of observations or of physical determinations. The constructional methods are in effect methods of *graphical computation,* and this constant checking of arithmetical and algebraic by geometric insight is the source of intense satisfaction, closely akin to that arising in the interaction between scientific theory and experimentation.

Arithmetical computations are so laborious that at present the graphical depiction of functional relations given by formula is of the

nature of a luxury, and I suppose everyone knows how little real significance attaches to the formula by itself. On the other hand, alert students and teachers will by various devices[1] execute with ease the graphical computations[2] of functional expressions.

7. For instruments use merely a straight edge in connection with the two systems of parallel linear scales furnished by the grating of the square-ruled paper, interpolating, if not by eye, by use of paper strip or dividers. But curves once graphically constructed are to be utilized as graphical tables; thus (*a*) the standard hyperbola $XY=1$ (Fig. 2) is a table of reciprocals Y of numbers X, and of reciprocals X of numbers Y; (*b*) the standard parabola $Y=X^2$ (Fig. 5) is a table of squares Y of numbers X, and of square roots X of numbers Y.

The graphical computation proceeds in scheme step by step parallel to the arithmetical computation. If a computation is impossible arithmetically—e. g., the extraction of the square root of a negative

[1]For instance, in Figs. 4 and 5 are indicated two constructions involving a straight line OLM, which appears in the various examples of each construction as OL_1M_1, OL_2M_2, OL_3M_3, etc. To effect these constructions expeditiously let the straight edge OLM of the ruler rest against a pin at O; then, in Fig. 5, since $UM=OB$, as M runs one by one along the marks of the scale UQ from U, the desired point L is marked on the consecutive vertical lines of the grating from the Y-axis, while, in Fig. 4, as L runs one by one along the marks of the line $Y_L=3$, with the eye one notes M on UP_0 and transfers by the eye, marking P on the vertical line through L.

Again, curves once constructed can be cut out and used later as rulers for the duplication of such curves perhaps in different positions with respect to the axes. The theory of such transformation of position of curves without change of form, and more generally the theory of transformations with or without change of form, and the corresponding theory of the transformation of the equations of the curves are of central importance.

Again, if a curve is known to be symmetrical with respect to some straight line or point, only one-half of the curve needs to be constructed directly.

Discriminate between the various curves of the same figure by letters for the general points of the respective curves (as below in this paper) and also graphically, by use of dots and dashes, shading, and colors, which should, of course, attach also to the elimination table and linkage diagram, which (as explained below) serve to epitomize the construction.

[2]I take the opportunity to refer to the article "Numerisches Rechnen," by Mehmke, in the *Encyklopädie der mathematischen Wissenschaften* (Leipzig: B. G. Teubner), Vol. I, article F, pp. 941–1081, for information as to the technical aspects of numerical computation, whether arithmetical, or by geometric drawing, or by the use of graphical tables (nomograms), or by apparatus, mechanical or physical. The nomographic methods are rapidly becoming of central importance. A nomogram once constructed for a certain type of problems of numerical computation, any particular

number—so is it graphically; cf. in §16 the construction for $\sqrt{1-x^2}$. If a computation is possible, but the result is not unique, as in the solution of equations, the arithmetical process usually requires tedious approximations, while the graphical process usually exhibits the results immediately in connection with the intersections of curves.

Of course, a computation of either type may be carried through in various ways; a comparison of these ways is interesting and instructive, and in the fundamental cases especially useful as serving to assist in determining a desirable set of geometric axioms.

8. In practice one handles addition directly by means of the linear scales of the paper. (Cf. §16.)

9. We propose the following problem in graphical multiplication. Designating in general the co-ordinates (X, Y) of a point K by X_K, Y_K, (to be read *the X of K, the Y of K*), having given a point P, we seek a point L such that to scale

$$X_L = X_P\,, \quad Y_L = X_P Y_P\,.$$

problem of the type is solved by the mere drawing of straight lines and noting the intersections of the lines with one another and with the curves of the nomogram.

Thus, to give a simple example, a nomogram for the solution of the general quadratic equation

$$X^2 + pX + q = 0$$

consists of the ordinary square-ruled paper with the standard parabola $Y = X^2$ (Fig. 5 below). The roots of any particular equation with given p, q are the X's of the points of section with the parabola of the straight line $Y = -pX - q$. This is the line crossing the X- and Y-axes in the respective points $(-\frac{q}{p}, 0)$, $(0, -q)$. One computes $\frac{q}{p}$ arithmetically or graphically (§9 below). Such an auxiliary computation is however unnecessary. Thus, d'Ocagne's nomogram (*loc. cit.*, §46) for the solution of

$$U^2 + pU + q = 0$$

consists merely of three curves (p), (q), (U) bearing three scales for p, q, U respectively, and the straight line joining the scale points p, q cuts the (U) curve in scale-points U, the roots of the equation to be solved. The three scales are, in terms of the square-ruled paper, as follows: the scale for p is the usual scale on the Y-axis; the scale for q is the usual scale on the line $X = 1$; the scale for U is $\left(\frac{1-X}{X}\right)$ on the hyperbola $Y = -\frac{(1-X)^2}{X}$.

Professor C. S. Slichter, of the University of Wisconsin, has arranged various styles of (uniform and non-uniform) ruling of cross-section paper (to be had of dealers in Madison) in such a way that all the elementary problems of logarithmic and trigonometric computation are to be solved merely by the drawing of straight lines.

This problem and naturally the corresponding problem in division: Given a point L, to determine the point P such that

$$X_P = X_L\ , \quad Y_P = \frac{Y_L}{X_L}\ ,$$

one solves by similar triangles using the figure $OUBLMP$, which appears in particular cases $OUB_1L_1M_1P_1$, $OUB_2L_2M_2P_2$, $OUB_3L_3M_3P_3$ in Fig. 4. We have in this figure in every case

$$X_M = OU = 1\ , \quad Y_M = UM = BP = Y_P\ , \quad X_L = OB = X_P\ ,$$

and, from similarity of the triangles OUM, OBL,

$$\frac{OU}{OB} = \frac{UM}{BL} = \frac{BP}{BL}\ ,$$

that is,

$$\frac{1}{X_P} = \frac{1}{X_L} = \frac{Y_P}{Y_L}\ ,$$

so that indeed

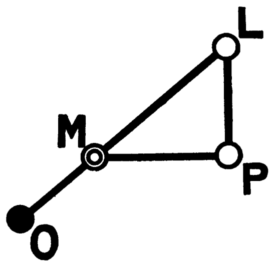
$$Y_P = \frac{Y_L}{X_L} \text{ and } Y_L = X_P Y_P\ .$$

It is to be noticed in the figure $OUB_3L_3M_3P_3$ how precisely the agreement or rule of signs in the multiplication of two signed numbers fits this geometric situation.

10. Geometric functionality comes to clearer vision by means of the notion of *linkage*—that is, an arrangement of links or bars with slots and pins, fixed or movable in slots, and pivots, and joints, and braces, such that, however the linkage be moved, certain essential properties of the diagram remain. The linkage represents geometrically the general variable diagram and algebraically a system of simultaneous equations between the simultaneous co-ordinates X, Y of the various points of the diagram, from which by elimination the relations sought are to be obtained.

In the *linkage diagrams* of this note points marked in black circles are fixed, those marked in white circles are free to move over the whole plane, those marked in two concentric circles are free to move each along a prescribed line or curve in the plane, while straight lines shown may move but so as to remain straight, and straight lines shown in the X- and Y-directions may move but so as to retain those directions.

Thus the linkage diagram and the *elimination table* for the multiplication figure discussed above are as follows:



$O=(0, 0)$

$X_M=1$

IV. **LINKAGE A**

	L	M	P
L		$\frac{Y_L}{X_L}=\frac{Y_M}{X_M}$ $\therefore Y_L=X_LY_M$	$X_L=X_P$ $\therefore Y_L=X_PY_P$ $\therefore Y_P=\frac{Y_L}{X_L}$
M	——	$X_M=1$	$Y_M=Y_P$
P	——	——	

IV. Elimination Table

An elimination table for a diagram involving the variable points L, M, P is an arrangement of six compartments LL, LM, LP, MM, MP, PP, where a compartment—e. g., MP—contains the entries connecting the points M and P, here the entry $Y_M=Y_P$. Here the entries preceded by the sign ($\therefore$) for "therefore" are obtained by elimination from the other four entries, which correspond directly to the conditions of the diagram. The entry MM, $X_M=1$, specifies the curve (M), in this case a line, on which M is to move. There are no entries LL, PP, because the points L, P are individually unconditioned and free to move over the whole plane.

11. Linkage A is a graphical multiplying machine: $Y_L = X_P Y_P$, and a graphical dividing machine: $Y_P = \frac{Y_L}{X_L}$. In any case of multiplication or division, if the two given numbers are not thus given graphically to scale as the co-ordinates (X, Y) of a point P or L, then a preliminary transformation of the data or a suitable modification of the process becomes necessary. Such a transformation would be by means of the lines of the grating and the standard diagonal $Y = X$, (OQ of Fig. 5 below) which throughout plays the rôle of a *turner* of X-segments into equal Y-segments, and *vice versa.*

For instance, if two points H and K are given, with $X_H = X_K$, the point G with

$$X_G = X_H = X_K, \quad Y_G = Y_H Y_K$$

is constructed by means of the figure involving linkage (A) and the turner (T), $Y_T = X_T$, specified by the following relations:

$$Y_H = Y_T = X_T = X_P, \quad Y_K = Y_P, \quad Y_L = Y_G, \quad X_G = X_H = X_K.$$

Construct linkage diagram and elimination table for this figure.

12. Clearly, if in linkage A the point P be made to move along some definite curve, call it the curve (P), the locus of the point P, corresponding to the algebraic relation[1]

$$(P): \; Y_P = \mathbf{P}(X_P); \quad \text{for instance, } Y_P = X_P,$$

between the co-ordinates X_p, Y_p, where $\mathbf{P}(X_p)$, (*the P function of the X of P*) denotes in general the algebraic expression in terms of

[1]As a result of the law of the motion of the point P, its co-ordinates X_P, Y_P will be always related in a certain way, expressed algebraically by the fact that Y_P always represents to scale a certain algebraic expression or function $\mathbf{P}(X_P)$ in terms of the co-ordinate X_P as taken to scale. We write for the curve (P),

$$(P): \; Y_P = \mathbf{P}(X_P),$$

or as another expression of the same meaning,

$$(P): \; X_P = \mathbf{P}^{-1}(Y_P).$$

These two equations are read: *the Y of P equals the P function of the X of P; the X of P equals the inverse P function of the Y of P.*

Every curve (P) determines in this way by the functional relations between the co-ordinates X, Y of its variable point P a pair of mutually inverse functions:

$$\mathbf{P}(Z), \quad \mathbf{P}^{-1}(Z),$$

and conversely every pair of inverse functions is representable in this way by a curve drawn to scale with respect to a co-ordinate system. The notations used are intended

X_P which is to scale equal to Y_P for every point P of the curve (P), then the point L will move along some curve (L),

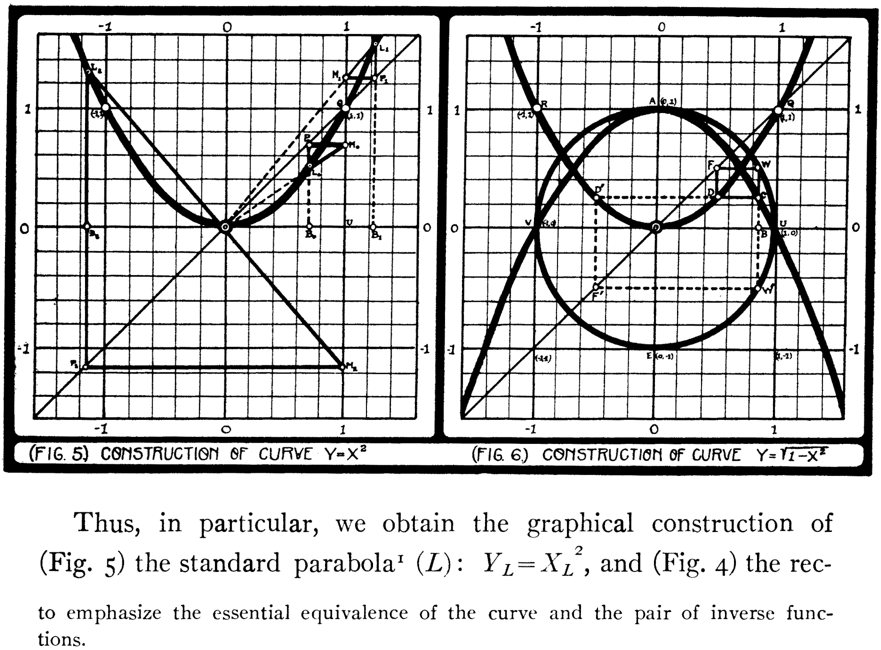
$$(L):\ Y_L = \mathbf{L}(X_L) = X_L \cdot \mathbf{P}(X_L)\ ;\quad \text{in particular, } Y_L = X_L^2\ .$$

Similarly, if L move along (L),

$$(L):\ Y_L = \mathbf{L}(X_L)\ ;\quad \text{for instance, } Y_L = 3\ ,$$

then P will move along (P),

$$(P):\ Y_P = \frac{\mathbf{L}(X_P)}{X_P}\ ;\quad \text{in particular, } Y_P = \frac{3}{X_P}\ .$$



(FIG. 5) CONSTRUCTION OF CURVE $Y=X^2$ (FIG. 6) CONSTRUCTION OF CURVE $Y=\sqrt{1-X^2}$

Thus, in particular, we obtain the graphical construction of (Fig. 5) the standard parabola[1] (L): $Y_L = X_L^2$, and (Fig. 4) the rec-

to emphasize the essential equivalence of the curve and the pair of inverse functions.

Given a curve (P) and a point Q; let the vertical line through the point Q meet the curve (P) in the point P_1; and let the horizontal line through the point Q meet the curve (P) in the point P_2. Then $X_Q = X_{P_1}$ and $Y_Q = Y_{P_2}$ so that

$$Y_{P_1} = \mathbf{P}(X_Q)\ ,\quad X_{P_2} = \mathbf{P}^{-1}(Y_Q)\ ,$$

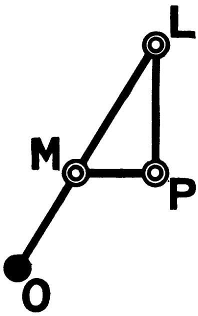
that is, *the Y of P_1 is the P function of the X of Q; the X of P_2 is the inverse P function of the Y of Q.*

Show how by use of the turner to compute graphically from the curve (P) and the point Q the functional expressions

$$\mathbf{P}(Y_Q)\ ,\quad \mathbf{P}^{-1}(X_Q)\ .$$

[1]Students of projective geometry will notice that the metrical constructions of Figs. 4 and 5 are precisely the Pascal hexagon projective constructions for conics,

tangular hyperbola (P): $X_P Y_P = 3$, with the following elimination tables and linkage diagrams:



V. **LINKAGE A′** (FIGS 5 AND 4)

$O=(0, 0)$	$O=(0, 0)$
$X_M = 1$	$X_M = 1$
$Y_P = X_P$	$Y_L = 3$
$\therefore Y_L = X_L^2$	$\therefore X_P Y_P = 3$
$(V$, Fig. 5)	$(V$, Fig. 4)

	L	M	P
L	$\therefore Y_L = X_L^2$	$Y_L X_M = X_L Y_M$	$X_L = X_P$
M	——	$X_M = 1$	$Y_M = Y_P$
P	——	——	$Y_P = X_P$

(V, FIG. 5) PARABOLA $Y_L = X_L^2$

	L	M	P
L	$Y_L = 3$	$Y_L = X_L Y_M$	$X_L = X_P$
M	——	$X_M = 1$	$Y_M = Y_P$
P	——	——	$\therefore X_P Y_P = 3$

(V, FIG. 4) HYPERBOLA $X_P Y_P = 3$

being given three points and tangent lines at two of the three points, viz. (Fig. 4), the point P_0 and the X- and Y-axes as asymptotes; (Fig. 5) the point Q and the X-axis and the line at infinity as tangents at their respective intersections with the Y-axis.

13. $\mathbf{L}(Z)=Z\cdot\mathbf{P}(Z)$.—This linkage A' yields, as stated above, more general curves for P, L, respectively, if L, P are made to traverse more general curves, viz.:

V. **LINKAGE A′ (GENERAL)** $\mathbf{L}(Z)=Z\cdot\mathbf{P}(Z)$

$O=(0, 0)$	$O=(0, 0)$
$X_M=1$	$X_M=1$
$Y_P=\mathbf{P}(X_P)$	$Y_L=\mathbf{L}(X_L)$
$\therefore Y_L=X_L\cdot\mathbf{P}(X_L)=\mathbf{L}(X_L)$	$\therefore Y_P=\dfrac{\mathbf{L}(X_P)}{X_P}=\mathbf{P}(X_P)$

For instance, if P traverses the quadratic parabola $Y=X^2$ of Fig. 5, L traverses the cubic parabola $Y=X^3$; if P traverses this parabola, L traverses the quartic parabola $Y=X^4$. Thus, by applying linkage A', being given the curve (P) and finding the curve (L), in succession starting from the standard diagonal $Y=X$ we obtain the curves $Y=X^2$, $Y=X^3$, $Y=X^4$, ... and, in general, $Y=X^n$ for n a positive integer.[1] The first quadrant of this curve serves for the whole curve, since the curve is symmetrical with respect to the Y-axis (n even) or the origin (n odd), and, it serves also, by interchanging the rôles of the X- and Y-axes, for the curve $X=Y^n$, that is, for the curve $Y=X^{\frac{1}{n}}$.

14. $\mathbf{G}(Z)=\mathbf{H}(Z)\cdot\mathbf{K}(Z)$.—Similarly, by means of the linkage of §11, we construct from two given curves (H), (K):

$$Y_H=\mathbf{H}(X_H)\ ,\quad Y_K=\mathbf{K}(X_K)\ ,$$

the curve (G):

$$Y_G=\mathbf{H}(X_G)\cdot\mathbf{K}(X_G)\ ,$$

viz., *the product as to Y of the two curves (H) (K), in the sense*

$$X_G=X_H=X_K\ ,\quad Y_G=Y_HY_K\ ,$$

so that

$$\mathbf{G}(Z)=\mathbf{H}(Z)\cdot\mathbf{K}(Z)\ ,$$

where Z denotes an arbitrary number the basis of the functional expressions **G**, **H**, **K**.

Of course, the same linkage otherwise employed yields from the curves (G) and (H) the curve (K) as the quotient as to Y of (G) by (H).

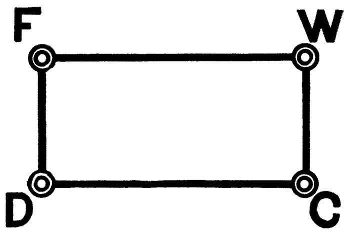
[1]The general case, n fractional, positive or negative, is treated in §21.

15. The linkage of §11 specializes to linkage A if $H=T$, and accordingly $K=P$, $G=L$. We have (cf. §13)

$$X_T=X_P=X_L\ ,\quad Y_L=X_PY_P=Y_TY_P\ ,$$

so that the curve (L) is the product as to Y of the curve (P) and the turner (T).

16. The rectangular linkage B (VI) is of central importance in the compounding of curves and functions. Three points C, D, F traverse given curves (C), (D), (F), and then W traverses a certain



$$Y_W=Y_F\ ,\quad X_F=X_D\ ,$$
$$Y_D=Y_C\ ,\quad X_C=X_W\ .$$

VI. **LINKAGE B**

curve (W).

Thus, if D traverses the turner $Y_D=X_D$ and F the straight line $Y_F=3\cdot X_F$ and C a curve $Y_C=\mathbf{C}(X_C)$, then

$$Y_W=Y_F=3\cdot X_F=3\cdot X_D=3\cdot Y_D=3\cdot Y_C=3\cdot\mathbf{C}(X_C)=3\cdot\mathbf{C}(X_W),$$

so that this use of linkage B effects the transformation of the curve (C) into a curve (W) as follows:

$$X_W=X_C\ ,\quad Y_W=3\cdot Y_W\ ,$$

and correspondingly constructs from a function $\mathbf{C}(Z)$ a function $\mathbf{W}(Z)$ where

$$\mathbf{W}(Z)=3\cdot\mathbf{C}(Z)\quad .$$

Similarly, arrange the linkage to construct from a function $\mathbf{P}(Z)$, graphically given by a curve (**P**) : $Y_P=\mathbf{P}(X_P)$, the various functions,

$$\mathbf{W}(Z)=2\cdot\mathbf{P}(Z)\ ,\ \tfrac{1}{2}\mathbf{P}(Z)\ ,\ \mathbf{P}(2Z)\ ,\ \mathbf{P}(\tfrac{1}{2}Z)\ ,$$

and

$$\mathbf{W}(Z)=\mathbf{P}(Z)+2\ ,\ \mathbf{P}(Z)-2\ ,\ \mathbf{P}(Z+2)\ ,\ \mathbf{P}(Z-2)\quad .$$

In these cases D traverses the turner and either C or F the curve (P), while the third point F or C traverses a straight line.

17. To apply linkage B to construct (Fig. 6) the circle

$$Y_W=\sqrt{1-X_W^2}$$

of radius 1 and center at origin, let F traverse the turner $Y=X$, and D the standard parabola $Y=X^2$, $X=\sqrt{Y}$, and C the parabola[1] $Y=1-X^2$, that is,

$$Y_F=X_F\,,\quad X_D=\sqrt{Y_D}\,,\quad Y_C=1-X_C^2\,,$$

so that indeed

$$Y_W=Y_F=X_F=X_D=\sqrt{Y_D}=\sqrt{Y_C}=\sqrt{1-X_C^2}=\sqrt{1-X_W^2}\;.$$

Construct this circle

$$Y_W=\sqrt{1-X_W^2}\,,\quad X_W^2+Y_W^2=1$$

by the same linkage also as follows:

$$X_D+Y_D=1\,,\quad Y_C=X_C^2\,,\quad X_F=Y_F^2\,;$$

so that indeed

$$1=X_D+Y_D=X_F+Y_C=Y_F^2+X_C^2=Y_W^2+X_W^2\;.$$

The curve (D) is in this case the straight line AU of Fig. 6.

18. $\mathbf{W}(Z)=\mathbf{FD}^{-1}\mathbf{C}(Z)$.[2]—The curve (W) is in a certain sense compounded out of the curves (C), (D), (F). Just so the two mutually inverse functions $\mathbf{W}(Z)$, $\mathbf{W}^{-1}(Z)$ associated with the curve (W) are compounded out of the functions associated with the curves (C), (D), (F). For, in the customary notations, with

$$Y_W=\mathbf{W}(X_W),$$

we have

$$\begin{aligned}Y_W=Y_F=\mathbf{F}(X_F)=\mathbf{F}(X_D)&=\mathbf{F}\big(\mathbf{D}^{-1}(Y_D)\big)\\&=\mathbf{F}\big(\mathbf{D}^{-1}(Y_C)\big)=\mathbf{F}\Big(\mathbf{D}^{-1}\big(\mathbf{C}(X_C)\big)\Big)\\&=\mathbf{F}\Big(\mathbf{D}^{-1}\big(\mathbf{C}(X_W)\big)\Big)\,,\end{aligned}$$

that is, in briefer form,

$$Y_W=\mathbf{W}(X_W)=\mathbf{FD}^{-1}\mathbf{C}(X_W)\,,$$

so that as operator on a general number Z the function $\mathbf{W}$ is the

[1]Obtained from $Y=X^2$ by reflection on X-axis and by lifting 1 in the direction of the Y-axis. Make this curve $Y=1-X^2$ also by use of linkage B as follows:

$$Y_F=1-X_F\,,\quad X_D=Y_D\,,\quad Y_C=X_C^2\,,$$

so that

$$Y_W=Y_F=1-X_F=1-X_D=1-Y_D=1-Y_C=1-X_C^2=1-X_W^2\;.$$

[2] To be read, the W function of Z equals the F function of the inverse D function of the C function of Z. This is the composition of functions effected in general by linkage B.

function resulting from the composition of the functions $\mathbf{C}$, $\mathbf{D}^{-1}$, $\mathbf{F}$ in that order, viz.

(a) $$\mathbf{W}(Z)=\mathbf{F}\mathbf{D}^{-1}\mathbf{C}(Z) .$$

Similarly,

(b) $$\mathbf{W}^{-1}(Z)=\mathbf{C}^{-1}\mathbf{D}\mathbf{F}^{-1}(Z) .$$

The formula (a) was just now derived from the linkage relations between the curves (C), (D), (F), (W) by process of algebraic elimination to express Y_W in terms of X_W.

It is desirable to see conversely how the linkage enables us to compute graphically by means of (C), (D), (F) for every number Z a corresponding number $\mathbf{W}(Z)=\mathbf{F}\ \mathbf{D}^{-1}\ \mathbf{C}(Z)$. Taking the Z as an X, say X_C, we have $\mathbf{C}(Z)$ given as

$$\mathbf{C}(Z)=\mathbf{C}(X_C)=Y_C ,$$

and then

$$\mathbf{D}^{-1}\mathbf{C}(Z)=\mathbf{D}^{-1}(Y_C)=\mathbf{D}^{-1}(Y_D)=X_D ,$$

and then

$$\mathbf{W}(Z)=\mathbf{F}\mathbf{D}^{-1}\mathbf{C}(Z)=\mathbf{F}(X_D)=\mathbf{F}(X_F)=Y_F .$$

Thus, to express Y_W in terms of X_W in the form $Y_W=\mathbf{F}\mathbf{D}^{-1}\mathbf{C}(X_W)$, we proceed around the rectangle in the sense $W\ F\ D\ C\ W$, and conversely, to compute graphically from given X_W the expression $\mathbf{F}\mathbf{D}^{-1}\mathbf{C}(X_W)$ as Y_W we proceed around the rectangle in the opposite sense $W\ C\ D\ F\ W$.

19. $\mathbf{W}(Z)=\mathbf{D}^{-1}(Z)$.—If C and F traverse the turner $Y=X$, then D and W are always symmetrical with respect to the turner, and

$$X_W=X_C=Y_C=Y_D , \quad Y_W=Y_F=X_F=X_D ,$$

so that linkage B transforms the curve (D),

$$(D):\ Y_D=\mathbf{D}(X_D) ,$$

into the curve (W) symmetrical to (D) with respect to the turner, viz.:

$$(W):\ X_W=\mathbf{D}(Y_W) ,\ \text{i. e.,}\ Y_W=\mathbf{D}^{-1}(X_W) .$$

Here

$$\mathbf{W}(Z)=\mathbf{D}^{-1}(Z) ,$$

so that this is a construction for the function $\mathbf{W}(Z)$ inverse to a given function $\mathbf{D}(Z)$.

The present relation $\mathbf{W}(Z)=\mathbf{D}^{-1}(Z)$ is in accord with the general

relation $\mathbf{W}(Z)=\mathbf{F}\,\mathbf{D}^{-1}\,\mathbf{C}(Z)$ of §17, since in the present case for every number Z $\mathbf{F}(Z)=Z$ and $\mathbf{C}(Z)=Z$.

20. $\mathbf{W}(Z)=\mathbf{FC}(Z)$ or $\mathbf{FD}^{-1}(Z)$ or $\mathbf{D}^{-1}\mathbf{C}(Z)$.—If D traverses the turner, so that $\mathbf{D}^{-1}(Z)=Z$, then linkage B constructs from given functions $\mathbf{C}(Z)$, $\mathbf{F}(Z)$ the compound function

$$\mathbf{W}(Z)=\mathbf{FC}(Z)\ .$$

Similarly, if C or F traverses the turner, we construct the compound function

$$\mathbf{W}(Z)=\mathbf{FD}^{-1}(Z) \text{ or } \mathbf{D}^{-1}\mathbf{C}(Z)\ .$$

21. The curve $Y=X^k$, associated with the function Z^k where $k=\frac{n}{d}$, n and d positive relatively prime integers, is constructed by linkage B from the curves:

$$(C)\quad Y_C=X_C^n\,;\qquad \mathbf{C}(Z)=Z^n\ ,$$

$$(D)\quad Y_D=X_D^d\,;\quad \mathbf{D}^{-1}(Z)=\frac{1}{Z^d}\ ,$$

which were constructed in §13, while F traverses the turner, for by §20

$$\mathbf{W}(Z)=\mathbf{D}^{-1}\mathbf{C}(Z)=(Z^n)^{\frac{1}{d}}=Z^{\frac{n}{d}}=Z^k\ ,\quad Y_W=\mathbf{W}(X_W)=X_W^k\ .$$

The curve $Y=X^{-k}=X^{-\frac{n}{d}}$ may be constructed from the same curves (C) (D) by letting F traverse the standard rectangular hyperbola

$$(F)\colon\ Y_F=\frac{1}{X_F}\ ;\quad \mathbf{F}(Z)=\frac{1}{Z}\ ,$$

(cf. §12, Fig. 4), for

$$\mathbf{W}(Z)=\mathbf{FD}^{-1}\mathbf{C}(Z)=\frac{1}{Z^{\frac{n}{d}}}=Z^{-\frac{n}{d}}=Z^{-k}\ ,$$

$$Y_W=\mathbf{W}(X_W)=X_W^{-k}\ .$$

Construct it also thus:

$$\mathbf{F}(Z)=\frac{1}{Z}\ ,\quad \mathbf{D}(Z)=Z\ ,\quad \mathbf{C}(Z)=Z^k\ ;$$

so that indeed

$$\mathbf{W}(Z)=\mathbf{FC}(Z)=\frac{1}{Z^k}=Z^{-k}\ .$$

22. The system of curves $Y=X^k$, $k=\pm\frac{n}{d}$, is worthy of remark. All the curves contain the point (1, 1) and run through the first quadrant $X>0$, $Y>0$ of the plane. We consider their distribution in the first quadrant. Those with k positive contain the origin (0, 0), those with k negative are asymptotic to both the co-ordinate axes. These two classes $(k=+)$, $(k=-)$ are separated by the curves $(k=0)$: $Y=1$, $(k=\infty)$: $X=1$, which partition the quadrant into four (call them) quarters; the class $(k=+)$ lies in the SW and NE quarters (to speak geographically as from the point (1, 1)), while the class $(k=-)$ lies in the SE and NW quarters of the quadrant. The curves $(k=\pm)$ fall into two classes $(k=\pm>)$, $(k=\pm<)$ according as k is numerically greater than or less than 1. The turner $(k=+1)$: $Y=X$ subdivides the class $(k=+)$, while the standard hyperbola $(k=-1)$: $Y=X^{-1}$ subdivides the class $(k=-)$. There are thus eight octants of the first quadrant. The curves $(k=+>)$ occupy the SSW and NNE octants, and come into the origin tangentially to the X-axis; the curves $(k=+<)$ occupy the WSW and ENE octants, and come into the origin tangentially to the Y-axis. The area inclosed by the turner, either co-ordinate axis, and the hyperbola $(k=-1)$ is in fact infinite. The curves $(k=-<)$ occupy the WNW and ESE octants, and inclose with the Y-axis a finite, and with the X-axis an infinite, area. The curves $(k=->)$ occupy the NNW and SSE octants, and inclose with the Y-axis an infinite, and with the X-axis a finite, area. These statements concerning areas are capable of elementary proof. The curves $Y_A=X_A^h, Y_B=X_B^{-h}$ are (cf. §14) as to Y reciprocals each of the other, since $X_A=X_B$ implies $Y_AY_B=1$; and they are like wise as to X reciprocals each of the other, since $Y_A=Y_B$ implies $X_AX_B=1$.

The march of the curves $Y=X^k$ in the other three quadrants of the plane depends upon the evenness—oddness character of the numerator and denominator of $k=\pm\frac{n}{d}$. The curves $\left(k=\frac{\text{even}}{\text{odd}}, \frac{\text{odd}}{\text{odd}}, \frac{\text{odd}}{\text{even}}\right)$ are symmetrical respectively as to the Y-axis, the origin, the X-axis, and pass from the first quadrant into the second, the third, the fourth quadrant.

23. By use of the curves $Y=X^k, Y=X^l$, each of the constructions (§17) of the circle by linkage B may be generalized, giving the curve

$$Y=\sqrt[l]{1-X^k}\ ,\quad X^k+Y^l=1\ ,$$

viz., in the first place,

$$Y_W=Y_F=X_F=X_D=\sqrt[l]{Y_D}=\sqrt[l]{Y_C}=\sqrt[l]{1-X_C^k}=\sqrt[l]{1-X_W^k}\ ,$$

and, in the second place,

$$1=X_D+Y_D=X_F+Y_C=Y_F^l+X_C^k=Y_W^l+X_W^k\ .$$

The second construction is the more symmetrical. In it the curve $X_W^k+Y_W^l=1$ arises by transformation via the curves

$$Y_C=X_C^k\ ,\quad X_F=Y_F^l$$

out of the straight line (*AU* of Fig. 6)

$$Y_D+X_D=1\ .$$

24. More generally, any curve

$$\mathbf{C}(X_W)+\mathbf{E}(Y_W)=1$$

arises by transformation via the curves

$$Y_C=\mathbf{C}(X_C)\ ,\quad X_F=\mathbf{E}(Y_F)=\mathbf{F}^{-1}(Y_F)$$

out of the straight line

$$Y_D+X_D=1\ ;$$

and similarly, in general, any curve

$$\Delta\big(\mathbf{C}(X_W)\ ,\ \mathbf{E}(Y_W)\big)=0$$

is constructed from the curve

$$\Delta(Y_D\ ,\ X_D)=0$$

by means of the curves

$$Y_C=\mathbf{C}(X_C)\ ,\quad X_F=\mathbf{E}(Y_F)\ ,$$

viz.;

$$0=\Delta(Y_D,\ X_D)=\Delta(Y_C,\ X_F)=\Delta\big(\mathbf{C}(X_C),\ \mathbf{E}(Y_F)\big)=\Delta\big(\mathbf{C}(X_W),\ \mathbf{E}(Y_W)\big)\ .$$

25. The reader perceives, or after a little practice will perceive, how readily the linkages A' and B alone or suitably combined serve to replace arithmetical by graphical computation of functional expressions,—and it is hoped that he will join with the writer in the following

Concluding Agreements

Today there is general agreement to cut out of arithmetic and algebra many complicated features of limited significance. We make the same agreement as to geometry.

In general, we agree to develop arithmetical, algebraic, geometric technique in a physical and intellectual environment logically and psychologically rich, full of movement, force, color, full of connotations and implications of and for real life of all kinds, including most certainly the real life of mathematics and the sciences.

To secure for our young students a tolerable appreciation of the civilization of the twentieth century on the side of the theoretical and applied mathematical sciences, as teachers of mathematics we agree to shape our instruction in mathematics from the beginning from a point of view no older and no lower than that of the wonderful seventeenth century, and to this end, speaking theologically, we propose to

Canonize the Cross-Section Paper.